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Mathematical Realism in Jean Ladrière and Xavier Zubiri: Comparative Analysis of the Philosophical Status of Mathematical Objects, Methods, and Truth

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Abstract

This paper analyzes and compares the philosophy of mathematics of Jean Ladrière and Xavier Zubiri. The study focuses on the status of mathematical objects and truth, the method proper to mathematics and finally the relationship between formal systems and the physical world. The philosophical context is the debate on the reality or ideality of mathematical objects and the four contemporary responses that dominated the 20th century: realism, naturalism, constructivism and conventionalism. These four responses face a series of limits and difficulties. Ladrière's transcendental realism and representational constructivism overcomes these difficulties. However, his position is characterized by a subtle dualism between mathematical reality, which exists independently of our intellectual efforts, and our mathematical representations. Zubiri's notion of sentient intelligence enables him to surpass the difficulties confronted by the four contemporary responses without yielding to dualism. Zubiri's philosophy of mathematics can be summarized with these two affirmations concerning mathematical reality: (i) it is not separated from our intellectual efforts; (ii) it is constructed according to concepts of sentient intelligence.

Resumen

El presente artículo analiza y compara las filosofías de las matemáticas de Jean Ladrière y Xavier Zubiri. Este estudio está enfocado en el estatuto filosófico de los objetos y de la verdad matemática, el método matemático, y finalmente a la relación entre los sistemas formales y el mundo físico. El contexto filosófico es el debate sobre la realidad o la idealidad de los objetos matemáticos y las cuatro respuestas contemporáneas que dominaron el siglo 20: el realismo, el naturalismo, el constructivismo y el convencionalismo. Estas cuatro respuestas presentan una serie de insuficiencias que ponen en duda sus tesis. Ladrière elabora una síntesis entre un realismo trascendental y un constructivismo de representaciones que supera dichas insuficiencias. Sin embargo, esta posición está caracterizada por un dualismo sutil que separa la realidad matemática, de la cual se afirma que existe independientemente de nuestros esfuerzos intelectuales, y nuestras representaciones matemáticas. La inteligencia sentiente, presentada por Zubiri, supera las insuficiencias de las cuatro respuestas contemporáneas sin caer en un dualismo. La filosofía de las matemáticas de Zubiri puede resumirse con las siguientes afirmaciones: (i) la realidad matemática no está separada de nuestra inteligencia; (ii) la misma es construida según conceptos de la inteligencia sentiente.

Introduction: the debate about the quintessence of mathematics

The period that comprises the end of 19th century and the 20th century witnessed the erosion of certainty in the fields of mathematics. In the past, this field was considered the model of rationality that made incontestable progress in objective knowledge. Nevertheless, this has changed due to a series of fundamental changes and debates in our understanding of formal sciences and their certitude. In more concrete terms, I present here the debate around the status of mathematical objects and methods in three mayor schools of philosophy of mathematics: logicism, formalism and intuitionism

A. Logicism

The basic idea of this school is that mathematical objects and properties can be defined from logical terminology and operators[1]. This idea followed the conjecture that arithmetic is an extension of logic. Frege sought to derive mathematical objective truths from logical truths. He affirmed that mathematical propositions have objective truth-values[2]. This view of objective truth-value expresses that mathematical affirmations are independent of language, minds, and conventions[3]. Frege developed a deductive system following definitions, logical rules and principles. For him, every truth about natural and real numbers is demonstrable following logical laws and definitions. Mathematical truths are a priori because they are not empirical facts[4].

Although Russell showed the serious limit of Frege's logicist program, he pursued the same objective of grounding mathematics in logic. Russell developed a version of logicism that does not deal with particular things or properties but with general and universal properties. During an initial period, Russell considered that numbers were classes, relations on classes, relations on relations on classes, etc. In his late writings, during the "no-class period", he took numbers and classes as logical fictions[5].

B. Formalism

1. Hilbert program

The objective of the *Hilbert program* was to "establish once and for all the certitude of mathematical methods"[6] and to guarantee the absolute objectivity of the intellectual efforts in mathematics. The conjecture at the background was that all problems could be solved[7]. According to this school, mathematics is an activity that operates over signs that do not have semantic content. For Hilbert, mathematical reality is identified with the concrete reality of signs. It is by means of the object-sign that we can go from the abstract to the concrete. Mathematical objects are nothing else than concrete signs[8]. For the formalist, definitions and rules are fundamental to mathematical method, which seeks to proof theorems. Consequently, mathematics becomes a body of demonstrable formulas. Mathematical truths are formal truths that depend on conventions, definitions and rules.

The objective of formalism is the axiomatization and formalization of various mathematical fields in order to ensure their coherence[9]. According to Hilbert, mathematics are formalized by providing (i) certain axioms that serve as building blocks for the formal structure of mathematics (axiomatization); (2) rules of deductions and construction. What is relevant in the deductive formal method is the set of axioms chosen and the rules. Intuition and observation are not part of the deductive process, although they could assist as heuristic. Axioms are functional definitions of mathematical objects and concepts. For that reason, it is decisive that they be consistent[10]. If a group of axioms is consistent then they are true and their defined objects exist. Therefore, mathematical existence is identified with the non-contradiction of the set of axioms. The rules of deduction are not arbitrary. They must enable the derivation of true propositions from consistent axioms[11]. A mathematical deduction eliminates all rational doubts by demonstrating that all theorems and mathematical truths are conclusions derived from the premises[12].

Hilbert's program, which sought to establish the certitude of mathematics, was deeply put in question by Gödel's incompleteness theorems.

2. GÖDEL'S INCOMPLETENESS THEOREMS

Let's consider Gödel's first incompleteness theorem[13]. This theorem affirms that in a formalized consist theory $\mathbf{F}_s$, there is a proposition $\mathbf{P}_s$ in the language of $\mathbf{F}_s$ where neither $\mathbf{P}_s$ nor its negation are theorems of $\mathbf{F}_s$. $\mathbf{P}_s$ is not provable in $\mathbf{F}_s$[14]. This is the case even if $\mathbf{P}_s$ is clearly true[15]. This calls into question that a single formal method can derive every arithmetic truth.

The first incompleteness theorem can be interpreted in two different ways. For a realist there is "more" than what is derived from the axioms[16]. Arithmetic cannot be reduced to deduction from the original axioms. However, there is a skeptical interpretation. According to this interpretation, the first incompleteness theorem states that some arithmetic propositions lack truth-values.

C. INTUITIONISM

For this third school, mathematics is primarily a mental activity. Mathematics exists in the human intellect. Mathematics is ground in a process of construction[17]. Brouwer argued that mathematical truths cannot be known by a mere analysis of mathematical concepts and their meaning. Although mathematics is a priori in the sense of being independent of empirical observations, it is dependent on the mind. Theorems could not be disproved empirically but they would not exist without the human mind. Brouwer, following Kant on this issue, proposed that mathematics is a mental construction. A proposition with a property Γ is established only if we show how to construct a number $\mathbf{n}$ that has the property Γ. For an intuitionist a mathematic object exists only if it can be constructed. That leads Brouwer to reject the notion that the law of the excluded middle[18] holds always a priori independently of a human construction. We do not dispose of an omniscient mind that can construct all mathematical propositions and their negations in order to assume that the law of the excluded middle always holds. Brouwer criticized logicism's statement that mathematics is an extension of logic and formalism's affirmation that mathematics is the practice of manipulating characters by following rules. For Brouwer the essence of mathematics was neither logic nor language. Language is just a medium to communicate the essence of mathematics: the mental construction[19]. Brouwer sustained that this mental construction of the mathematical edifice is grounded in a primordial intuition. Intuition is a way of knowing that is neither abstraction nor analogy. The primordial intuition is a direct insight, an a priori introspection in the individual mind leading to the awareness of time and mathematical construction[20]. Finally, the objective of intuitionism consists in grounding non-constructive mathematics in a constructive foundation. This objective was also put in question by Gödel's incompleteness theorems. According to Gödel's first theorem there is a mathematical proposition that is not provable in strictly constructive principles[21].

Table 1 summarizes some philosophical positions of the three schools already discussed.

II. A typology of the status of mathematical objects:

The previous section provides a brief historical background about the philosophical debate regarding the essence of mathematics. Beside that debate, there is another discussion among philosophers and mathematicians about the status of mathematical objects[22]. Do they exist? How do we have access to them? This section sketches different positions regarding the philosophical status of mathematical reality. For pedagogical reasons, I group these positions in four major types or models[23]. These four types are: realism, naturalism, constructivism and conventionalism. Let us consider these positions.

	Logicism	Formalism	Intuitionism
Project	To ground mathematics' certainty on logic.	To establish the certitude of mathematics on deductive axiomatic method.	To ground mathematics' certainty on constructive bases.
Essence of mathematics	*Logic* Mathematical truths are logical truths.	*Language* Truth is reduced to deduction from consistent axioms.	*Mental construction* grounded on the intuition and finite operations.

Table 1. Some philosophical elements of logicism, formalism and intuitionism

A. Realism

Mathematical realism considers that mathematical objects have an objective reality and existence independently of the mathematician's mind, language and convention. For some realists, sometimes called Platonist realist, mathematics refers to an eternal, unchanging and ideal realm which is not part of space-time[24]. A definition does not construct an object. Rather it points to an existing object[25]. Frege and Gödel ere two figures that sustained this position.

Frege believed that natural numbers exists independently of the mathematician mind[26]. They are not subjective product of the intellect. Gödel disagreed with the idea that mathematical objects are constructed out of definitions, concepts or attributes. Gödel affirmed that "*we form our ideas also of those objects on the basis of something else which is immediately given*"[27]. Gödel did not say what was this "something". For him, we have access to objective mathematical objects due to a mathematical intuition, analogous to sense perception, which leads to mathematical knowledge[28]. By means of this intuition, some mathematical principles "force themselves on us as being true". Mathematical objects exist independently of our constructions and we access them by means of mathematical intuition. However, mathematical intuition is fallible and can lead us to paradoxes (such as Russell's paradox). To understand and grasp the properties of mathematical objects we have to go beyond mathematical intuition and axiomatic descriptions. Axioms and mathematical intuitions do not contain a complete description of the mathematical reality, which is a consequence to the first incomplete theorem[29]. In order to proof some simple propositions in elementary mathematics we have to go to richer theories, e.g., real analysis and set theory.

Finally, realists affirm that mathematical truths are a priori and necessary truths. Mathematical propositions are not contingent as scientific propositions. They are prior and independent of any observable experience. Mathematics truths are necessary because they could not be otherwise[30].

In spite of many efforts to address the issue, mathematical realism still has difficulties in explaining how we, physical realities, have access to "real" mathematical objects that exist in a mathematical realm independently of our minds.

B. Naturalism[31]

A naturalist challenges the idea that a physical being in a physical universe has access to a mathematical realm detached of his reality[32]. There is no a priori truth grounded in concepts and meaning independent of facts. Quine, a representative of this position, accepts that some propositions are true in virtue of definitions, concepts and meaning. However, for him, this is not the central aspect of scientific knowledge. The only evidence relevant to a theory is sensory evidence. What exist is concrete or physical. Mathematics is important and legitimate only to the extend that it aids science empirical sciences[33]. Mathematics has a central place in the process of understanding our physical universe and has the same status as the most theoretical aspects of science. Nevertheless, it is within science that reality is to be identified.

Quine's position affirms an epistemology of objective truth-value. Quine affirms that sciences, including mathematics, do not look for extra-scientific criteria to judge mathematical or scientific truth. Scientific and mathematical truths are *a posteriori* objective truths grounded in empirical experiences[34]. Consequently, the application to our concrete physical universe is a criterion of mathematical truth (pragmatism). However, there is a problem. As a matter of fact, mathematicians in their practice do not depend on mathematical applications to ground and verify mathematical truth.

Mathematical naturalism experiences a major difficulty when it tries to justify highly abstract branches in mathematics that are developed without any empirical reference (set theory, abstract algebra, etc.).

C. Constructivism.

A constructivist argues that mathematical objects exist but as free creations of the human spirit[35]. He does not accept that mathematical propositions are true or false independently of the mathematician's mental activity. Truth and false must be understood in a constructive sense[36]. The mathematician must show that there exists a method that enables the generation of the mathematical object. An intuitionist argues that there is no criterion of truth independent of the construction process in the human mind[37]. Consequently, a constructivist does not agree with truth-value realism[38].

The constructivist must address the challenge rise by the fact that our finite mind can produce infinite objects (number, functions, etc)[39].

D. Conventionalism

For conventionalism mathematical objects are pure linguistic constructions[40]. A conventionalist sustains that mathematical language does not have real and existing reference[41]. Russell affirms that mathematics can be reformulated in terms of properties and concepts with no reference to mathematical objects such as numbers, functions, classes, etc. This is Russell's position during his "no-class" period. For him the mathematical objects are nothing else than logical fictions with a correct linguistic application. The introduction of a frame of reference and units is nothing else than an arbitrary convention. Thinking about mathematical reference, Putnam wrote "[…] reference itself begins to seem "occult"; that it begins to seem that one cannot be any kind realist without being a believer in non-natural mental power"[42].

In terms of mathematical truth there are different positions. Some conventionalists, such as Russell, believe in the objective truth-value of mathematical propositions. Others, such as Hartry Fields, sustain that the truth-value of mathematics is vacuous since mathematical object does not exists. The proposition "all natural number are prime" lacks truth-value because natural numbers do not exist.

Conventionalism faces the difficulty of explaining the successful application of mathematics to the physical universe. How does a mathematical theorem, without any reference, tell us something about the natural world and our human economic actions?

Table 2 summarizes the four models, their conceptions of mathematical objects and truth, and their difficulties.

Models	Object	Status of mathematical truth	Difficulty
Realism (Gödel and Frege)	Exists independent of the mathematician.	A priori and necessary truths. Objective truth-value	Justify access to the mathematical reality.
Naturalism (Quine)	Only concrete and physical objects exist.	Neither a priori and necessary nor purely empirical. Objective truth-value.	Justify abstract mathematics without relationship with concrete objects.
Constructivism (Brouwer)	A mathematical object exists if it can be constructed.	A priori, independent of observations. Not necessary, they depend on construction.	Justify that our finite mind produces infinite objects.
Conventionalism (Russell and Putnam)	Mathematical objects are pure fictions. Mathematics does not make reference to existent mathematical objects.	Some sustain objective truth-value. Others think that truth-value of mathematics is vacuous.	Explain the successful application of math, based on conventions, into the physical universe.

Table 2. Four models concerning the philosophical status of mathematical objects.

III. Jean Ladrière's transcendental realism and representational constructivism

After presenting this historical background, let's considers the mathematical philosophy of Jean Ladrière[43] and his perspectives on the status of mathematical objects, method, truth and relationship with the physical world. Ladrière develops his reflection around the formal axiomatic systems, their coherence and their limits. He seeks to understand the ground of mathematics, its rational project, and objectives[44]. By presenting his philosophy of mathematics, we will see how Ladrière answers the four difficulties found in the four models concerning the status of mathematical objects.

A. The status of the mathematical objects:

Ladrière elaborates a synthesis on some aspects from realism and constructivism. For him, the mathematical object is characterized, at the same time, by being constructed and given before the mathematical reflection[45]. Influenced by Gödel, Ladrière affirms that the mathematical objects exist already before all intellectual activity. In what could be considered a platonic position, he states that these objects are ideal. Consequently, mathematics explores a realm that is already constituted. At the same time, he also argues that mathematics provides itself its own objects and their existence by means of definitions and axioms[46]. The mathematical object manifests itself progressively through the history of mathematics. This manifestation occurs due to a dynamical construction of necessary symbolism, which constitutes a new language achieved in formalism[47].

Does that mean that Ladrière hesitate unsure between realism and constructivism? In order to see how Ladrière clarifies his positions, we must understand his analysis of the mathematical axiomatic method. Ladrière establishes a clear principle: an object cannot be understood without referring it to a particular rational method. An object is not a pure reference to itself. Its meaning is found in relation to the objective and method of mathematics[48].

B. Formal systems and the axiomatic method:

Ladrière considers that a formal system is an ideal system constituted by a group of theorems, which are derived from axioms following a set of rules[49]. An axiomatic formal system consists of a group of conventions that determine a set of objects, a set of propositions and a set of theorems. The set of objects consists of a collection of elementary objects, also named elementary symbols or atoms, and of certain number of operations that permits the construction of complex objects from atoms. An operation is a transformation that changes an entity into another entity, e.g. the arithmetic operations of addition, subtraction, multiplication, and division.

It consists of elementary objects, operations, predicates, formation rules, a set of axioms, and rules of deduction. Consider the following example, the axiomatic formal system of natural numbers as expressed by Peano.[50] The example is shown from table 3 to table 5.

Elements of a formal system	*System of natural numbers*
Elementary symbols	The elementary object: "0".
Operations	One operation: "*S*". For all "*x*" in the system, *Sx* is the successor of *x*.
Formation rules. A grammar that tells how formulas or propositions are to be constructed.	Predicate: "=" If *x* and *y* belong to the system, *x* = *y* is a proposition of the system.

Table 3. Morphological components: elementary symbols, operations and rules of formation in a formal system.

This first part refers to the morphological components. Here we find concepts and symbols (also named elementary objects or atoms) are explicitly introduced. First, there is the explicit list of elementary primitive components. In this example, it is "0". Then there is a list of operations that operate over the elementary symbols. Finally, there are formation rules that, following a set of predicates, form the propositions of the system from elementary objects.
The second section is the axiomatic part. It consists of a set of axioms and rules of deduction. Axioms are propositions from which it is possible to derive theorems by following rules of deduction[51].

Set of axioms	(i) There is a natural number "0" (ii) There is no natural number whose successor is 0. (ii) If *x* is a natural number, then the successor *Sx* is also a natural number. (iv) Distinct natural numbers have distinct successors. If *x* ≠ *y*, then *Sx* ≠ *Sy*. (v) If a property is possessed by 0 and also by the successor of every natural number, then it is possessed by all natural numbers. (principle of induction).
Rules of deductions	If *x* = *y*, then *Sx* = *Sy*

Table 4. Axiomatic part: set of axioms and rules of deduction

Axioms have definitions that are in some way arbitrary and are presented as valid. The axioms are chosen freely. The only criterion for choosing these axioms is internal coherence. These axioms fix the meaning of propositions. The formulation of deductive rules eliminates all recourse to intuition[52].

Table 5 shows some examples of derived propositions.

Examples of derived propositions	0 = 0; *S*0 = *S*0; *S S*0 = *S S*0, etc. General proposition: If "*y*" is an object of our system, *y* = *y*.

Table 5. Examples of derived propositions in natural numbers.

Analyzing the axiomatic method, Ladrière states in an article of 1966, that there seems to be a paradox. Mathematics explores a domain that is unknown; we do not know in advance the properties of the mathematical objects. At the same time, mathematics provides to itself its own objects by doing some construction and creation[53]. Consequently, mathematics is partially constructed and partially given. Well, this paradox turn to be apparent. Ladrière solves this paradox by distinguishing between the mathematical object and its representation. There is a duality between mathematical reality and its representation in a mathematical language. Let's consider these two aspects and their relation.

1. Priority of mathematical reality

Ladrière proposes the following thesis: *mathematical reality is objective and is autonomous from the mathematical method. Mathematical objects are accessible by means of mathematical representations. These representations constitute a mathematical language that enables the object's concrete manifestation[54]. Therefore, mathematical reality is before and beyond the formal language.*[55]

According to Ladrière, three arguments sustain this thesis: the history of mathematics, the plurality of axiomatic approaches and the inadequacy of axiomatic systems.[56] The argument of the history of mathematics is as follows. There is a historical process, marked by contingency and intuitions, in which we discover and grasp mathematical objects and their understanding. However, once we grasp these objects, we leave behind this contingent process and express it in a formal system. This formal system expresses and grounds what was known before[57]. The second argument, the plurality of axiomatic approaches, shows that the same mathematical objects can be represented through different axiomatic systems. The same object is delimitated by different definitions that belong to different axiomatic processes.[58] The third argument is the inadequacy of the axiomatic systems. Ladrière interprets Gödel's theorem of incompleteness as an evidence of the insufficiency of axiomatic systems' representation of mathematical reality.[59] Therefore, mathematical reality exceeds all intellective and linguistic effort. Mathematical reality is autonomous with respect to our intellectual effort.

2. Mathematical representation as the conditions of the possibility to apprehend the mathematical reality.

Language enables the mathematical formal object to obtain, metaphorically speaking, a body to manifest itself.[60] The axiomatic method is a valuable instrument for the study of mathematical reality. It enables us to grasp and delimitate the object. The formalization helps us have access to the mathematical formal objects through an experience that is neither an immediate intuition of the mathematical reality (against intuitionism) nor an empirical experience (against naturalism).[61] The choice of criteria, rules, operations and axioms determine the type of object that will be manifested. In other words, our access to a particular type of object depends on particular choices we made at the internal structure of formal languages. These choices (axioms, rules, definitions, operators) do not determine the internal structure of the object, but our possible access to it. As a consequence, the method is proportioned to the nature of the object. The diversity of axiomatic methods corresponds to the diversity of mathematical objects.[62] The apparent subjectivity in the choice of criteria, axioms, operators and rules is really an adaptation of the method to the rigorous requirements of the object itself. Mathematical axiomatic formal language should not be understood as a creation of an object, but as the development of an access to it.[63]

Gödel theorem is interpreted as follows: the linguistic manifestation is partial, historical and never total[64]. There is a horizon of mathematical reality that is always open and never fully apprehended in the formalization project. However mathematical rationality is partial, historical and constantly becoming. Gödel's theorems show that reason is not always victorious, master of the world and of itself. On the contrary, it is a humble effort always uncertain, discontinuous, limited and in development. It is always in need to integrate its own failures.[65]

C. Mathematics relations to the physical world.

Against a platonic conception, Ladrière sustains that our access to the mathematical realm is constituted initially through perceptive experiences. There is a double movement of going from the perceptive world to the mathematical realm and returning back. This double movement explains the fact that mathematics is an efficient instrument in the knowledge of the physical universe. The relationship between the mathematical object and the physical world happens at the genesis of the object's representation and at its utilization.

1. The genesis of the object's representation: a movement from the concrete to the general and abstract.

The development of mathematical representation is a historical process that depends on the relationship to the sensible world by means of three steps: schematization, thematization and the abstraction of general structures.[66]

(i) Schematization. By starting from the experience of concrete objects and their complexity we extract progressively a schema by which we substitute a perceived object with a formal object. At this step there is a clear relationship between mathematics and perception.

(ii) Thematization. From different levels of abstraction, we develop more abstract theories. At this step there is no more direct relationship to the perceptive physical world.

(iii) General abstract structures. Finally, we group a series of theories that are alike in order to develop more abstract and general domains of objects.[67]

Concrete objects and physical situations or problems suggest mathematical objects and theories.[68] This is a movement from perceived and sensible objects, forms and structures to abstract objects and structures. Nevertheless, mathematical objects do not have the same status as concrete physical objects[69]. In the genesis of the object's representation, once we arrive to the step of thematization, the mathematical object is autonomous and has a priority over its process of genesis. There is no more reference to the sensible world. Consequently, mathematics cannot be reduced to physics. At the same time, Ladrière critics the platonic vision that mathematical objects subsist in themselves as floating things. He sustains that mathematical reality is autonomous from the sensible world, but is neither an independent nor autarchic reality from the physical world. Applications show there is a mediation between both, the mathematical reality and the physical reality.[70]

2. Applications: movement from the abstract to the concrete.

An application is a movement from abstract structures and objects to concrete objects, structures and situations. Although mathematical reality is autonomous with respect to the physical reality, the genesis and the applications of mathematics shows there is a mediation between the abstract mathematical realm and the physical world. Ladrière states that this mediation is characterized by our "interpretation".[71] Interpretation is a relation between the propositions of the formal system and their mathematical objects with other mathematical disciples. For example, we can relate arithmetic's objects and propositions with algebraic structures and set theory objects. Another example is the relationship developed by Descartes between geometry and algebra. There is another type of interpretation: the relationship between mathematical systems and physical systems. For example, the mathematical expression $F = m\frac{d^2x}{dt^2} = ma$ can be interpreted in terms of its application to a mechanical system: *force is mass times the acceleration of an object*. Interpretation enables the formal knowledge to enter into the domain of phenomena, which is the domain of the empirical experiments.

The application of mathematics is an interpretation of the mathematical formal system and at the same time it is a hypothetical representation of a physical process or phenomenon we want to understand. In this hypothesis we have to choose between different formal systems because some applies better than others, and some do not apply at all. A scientist has to verify that a particular system represents well a physical reality.[72]

D. Mathematical truth and the criterion of truth:

Ladrière's analysis of the axiomatic formal method makes him conclude that mathematical reality and truth are objective. Mathematical truth is independent of our intellectual effort.[73] However, we can never remove completely the uncertainties, shadows and opacity that surrounds our apprehension of mathematical objects and truths due to the fact that we cannot fully represent mathematical reality and its truth in a closed formal system.

Although Ladrière is not explicit about the criteria of truth in formal sciences, we can understand it thanks to the notion of interpretation. The criterion of correspondence is applied here because a formal system is "verified" in its relationships with other formal systems. It can also be verified with empirical phenomena, such as mechanical processes. The criterion of coherence is used to confirm that a particular proposition is "coherent" with the rules of deduction and with other propositions, including axioms of the formal system. The fact that a formal system would not have longtime interest if it is not interpreted, implies a version of the pragmatic criterion of truth: the utility a formal systems in terms of its use to interpret other formal system and the domain of empirical phenomena. Even highly abstract mathematical systems can become an instrument for scientific development such as non-Euclidean Riemann's geometry, which inspired Einstein's General Theory of Relativity, and Hilbert's space, which enabled Quantum Mechanics. These examples imply that formal systems are open to be applied to the domain of empirical phenomena.

III. Xavier Zubiri: mathematical reality as constructed by postulation

Let us consider in this section Zubiri's mathematical philosophy and his perspectives on the status of mathematical objects, method and truth. His reflection on the subject is grounded in his notion of sentient intelligent that enables him to overcome many difficulties encountered in the classical models, while avoiding Ladrière's dualism between mathematical reality and our representation of it.[74]

A. Mathematical objects

For Zubiri, mathematical objects have reality before intelligence.[75] However, it is not a reality that subsists by itself, but a postulated reality.[76] Mathematical objects are not only apprehended, they are constructed by the intelligence and they have a reality by postulation. According to Zubiri, there are two types of real things. First, there are things that are real in and by themselves, e.g., a stone, a tree, an animal, a human being. Second, there are things that are made real by means of an intellectual construction according to concepts.[77] Mathematical objects correspond to the second type.[78] The content and the mode of reality are different: the stone, which is a perceptive reality, is real in and by itself while the circle has a reality by postulation. The reality of a mathematical object, is placed by a double act: (i) a definition of that reality, and (ii) a postulation of its reality.[79] Therefore, mathematical realities are realities defined and postulated.

Zubiri disagrees with mathematical transcendental realism, because he denies that mathematical objects are in and by themselves real. He also disagrees with formalism and its notion that mathematics is grounded on language. Mathematics is neither a system nor a language defined by operations, concepts and rules. Zubiri also disagrees with Brouwer's intuitionism and his position that the foundation of mathematics is a series of executed operations and the data of intuition. Finite or infinite sets are not formally intuitive. According to Zubiri, we do not have an intuition (immediate, direct, unitary vision of something) of a set of elements. A set of elements is the results of an act of construction in and by the intellect. It is the application of the concept of set, concept already constructed in the intellect, to the diversity of the given[80]. Zubiri also criticizes Brouwer is his affirmation that the essence of construction is the execution of a series of objective operations. Brouwer's sets and operations are objective concepts (conceptives). Mathematics is not about objective concepts, it is about "things which are thus".

A mathematical object is neither real by a mere definition nor by the execution of a series of operations. A mathematical object is real by a postulation that realized a content (properties and existence) freely determined[81]. The mode of intellection is not a mere conceptuation, in the sense of idealization, but a realization. The existence and the properties of a mathematical object are freely postulated. It is a real object constructed according to concepts.

In order to understand Zubiri's notions of realization as construction according to concepts, we need to understand his conception of mathematical method.

B. Mathematical method: construction according to concepts.

Formalists affirm that deduction is the heart of mathematical method. According to Zubiri, that is not the case. Deduction is not a method, but part of the logical structure of mathematics. Mathematical judgments and the logical structure of the reasoning process are not formally a method, but something the mathematical structure must respect. It is not enough to define rules of deductions; we have to "make" the deduction by operating, transforming and constructing within mathematical reality[82]. Mathematical method is not the means by which we reach mathematical reality, as affirmed by Ladrière. The method is already installed in the mathematical reality by the process of postulation. The mathematical method moves in reality itself[83].

Zubiri states that a method is forging a way in order to deepen reality itself, whether given or postulated reality. In all methods there are three moments: (a) the establishment of a system of reference, (b) the sketch and (c) the experience or testing. These three moments are not independent or purely sequential. Each one recovers the other. Let's consider each of these moment in the mathematical method.

1. THE ESTABLISHMENT OF A SYSTEM OF REFERENCE.

Mathematical reality has been postulated by suggestion of field reality because reason already moves in field reality qua reality[84]. Mathematical reality is not synonym of field reality[85]. The field reality is converted into a system of reference by which mathematical reality has a content. At the same time, this content follows a suggestion in the field reality. Nonetheless, the field reality (acting as a suggestion) and the system of reference (which enables the content of the mathematical reality) are not identical[86]. Let's consider an example, the relationship between perceptive space (field reality) and geometric space (system of reference). The perceptive space or field space has some real characteristics. Reason converts this perceptive field or space into a geometric space (Euclidean space, Hilbert's space, Riemann's, or another), which is a system of reference. Geometry consists of a free system of axioms, definitions, rules of constructions, and rules of deductions that postulates the precise content of the geometric space[87]. Geometric reality is a postulated reality, a reality constructed according to geometric concepts. Although the perceptive field space suggests some elements to sketch a geometrical space, they are not identical. The content of the geometric space (real characteristics postulated by axioms, definitions and rules) is different of the perceptive pre-geometric space.

Every mathematical object is defined and postulated in this system of reference to which it belongs[88]. The reality of a mathematical object is apprehended in reference to this system of reference[89]. It is the whole system of reference which is defined and postulated. The mathematical realm is not a juxtaposition of independent mathematical objects. Every mathematical object has meaning and reality in a particular system of reference.

2. The formal terminus of the mathematical method: the sketch

The sketch is precisely what is constructed by the conversion of a field reality into a system of reference[90]. This sketch of reality is a mathematical construction. The content of the sketch is freely constructed according to concepts. As Zubiri affirms:

> The mathematical object is not constituted by the postulates; rather, what the postulates define is the "construction" before the intelligence of that whose realization is postulated, and which acquires reality by this postulation.[91]

This is not a construction of objective concepts, but the construction of mathematical object through concepts[92]. Mathematical objects have properties that are defined through axioms and other mathematical concepts[93]. However, there are more properties *de suyo* in this object than those defined.

The sketch relies in the suggestion coming from the field reality and at the same time is independent from it. Its independence enables a free creation of the whole content, which implies that reason constructs the properties of objects and their basic structure[94]. In Zubiri's words:

> Although my free construction adopts models or basic structures taken from the field, nonetheless the free construction is not formally constituted by what it adopts; if it does adopt it, it does so freely.[95]

This free construction consists in creating a content with full freedom by postulation. We "sketch a free system of axioms" that determines the content of reality[96]. These axioms are not truths that I freely state by means of linguistic affirmations. Axioms are about real characters that I freely sketch[97].

What is the role of conclusion and deduction in the sketch? In any deduction, the conclusion has two moments that are inseparable but different. The first moment, called the moment of necessary truth of a mathematical judgment, refers to a conclusion that follows necessarily from axioms, definitions and rules of deductions. In the second moment, the moment of apprehension of reality, when I affirm that A "is" B, I am affirming not merely a truth, but a real property of a mathematical object: A "is really" B. Consequently, mathematics is not a pure logic about truths, but it is a science about reality[98]. The moment of apprehension of reality precedes the moment of necessary truth because mathematical axioms are about real characters and not truths[99]. Reality precedes logical judgments[100]. The intrinsic unity of these two moments, necessary truth and apprehension of reality, is what constitutes the experience as "*comprobación*", which is translated as testing-together[101].

3. Method as experience[102]: "comprobación" as testing-together.

According to Zubiri, this is a mode of mathematical experience that tests postulated realities. What we are testing or verifying here is not the truth of a mathematical affirmation, but the verification of mathematical reality in its truth. Mathematical methods leads us to apprehend the reality of A "as being" B[103]. By means of mathematical experience, I am testing together the reality of A and the reality of B in the formula A "as being" B[104].

C. Mathematical truth and verification

What is postulates is real before is true. Therefore, mathematical reality precedes mathematical truth. Zubiri interprets Gödel's theorem as follows: *what is constructed by postulation has "de suyo" more properties than the properties formally postulated*[105]. After distinguishing between mathematical truths and mathematical reality, Zubiri asks if mathematics truths are exact or an approximation of mathematical reality? Zubiri concludes that mathematical truths are not completely adequate to mathematical reality. The adequation between mathematical judgments and the mathematical reality is only a remote goal of a never-ending dynamism of the intellect. All true judgments, which are in conformity with reality, point to this unreachable far-off goal of adequation[106]. Consequently, all mathematical judgments that are true are structural approximations to what should be a truth adequate to reality. Approximation here does not mean inaccuracy, falsehood or deficiency. It means gradual. Mathematical judgments are truth and necessary, but they approximate the whole mathematical reality. We are not capable, in a mathematical judgment, to apprehend the whole reality of a mathematical object. Conformity can become more and more adequate in a dynamic and historical process[107]. As Zubiri affirms, if the mathematical reality:

> …had no structural properties other than those defined and postulated, every mathematical judgment would be true in the sense of being just an aspect, and therefore everything defined and postulated would be adequately apprehended in each thing. But this is not the case.[108].

This follows from his interpretation of Gödel's theorem. In his own words:

> Gödel's theorem shows that the whole thus postulated and defined necessarily has properties which go beyond what was defined and postulated. This definition and these postulates in fact pose questions which are not resolvable with them alone. And therefore these solutions are just the discovery of properties which go beyond what was defined and postulated. Then the adequate intellection of each thing in this whole is left, at each step, outside of what was defined and postulated, properties which intellective movement does not achieve. These properties are not just "more" definitions and postulates, but rather are necessary properties of the thing and confer upon its reality a distinct structure in the complete whole.[109]

Against Leibniz, Zubiri rejects the notion that mathematical truths are eternal truths grounded in concepts. Mathematical truths are necessary, but their necessity is grounded on the reality as given in and by the postulates[110].

D. Relationship between mathematics and physical world

An analysis of Zubiri's thought shows that for him the relationship between mathematics and the physical world depends on three postulates intimately connected: (i) the postulate that cosmic reality has a mathematical structure[111]; (ii) the creation of a content of a sketch; (iii) then the postulate that this sketch corresponds to a particular cosmic reality. The first one is the postulate that science has followed since the success of Galileo. The second is suggested by a field reality and constructed according to concepts. The final postulate refers to the structure of scientific hypothesis. Of course, there are many possible systems of references and sketches. Part of the scientific method is to choose one sketch among all possible systems of reference and sketches and testing it. In this point Ladrière and Zubiri coincide.

IV. Conclusion

Table 6 summarizes the major positions of Ladrière and Zubiri regarding some issues and debates in philosophy of mathematics. An initial reading shows many similarities and proximities. Both agree in the priority of mathematical reality over language and their affirmation that mathematical reality exceeds mathematical judgments. They sustain the idea that perceived reality suggests mathematical abstraction, although mathematical reality is autonomous. They also agree also that mathematics is always open to be applied to physical realities, structures and situations.

However, there are some major differences. Ladrière sustains a dualism between mathematical reality and the mathematical symbolic language. The first is given while the seconfd is constructed. In the background, Ladrière conceives mathematics reality in terms of concipient intelligence; as something given to the intelligence. Mathematicians conceptualize mathematical reality by means of mathematical symbols, defined concepts, rules and axioms of truth. For Ladrière, the essence of mathematics is language. This is what Zubiri calls the *logification of intellection*[112], which is the classical view that subsumed intellection under the *logos*.

Zubiri avoid this dualism thanks to his notion of sentient intelligence. Mathematical reality is not a realm separated from our intellect. We are already in this postulated reality constructed according to concepts and this is precisely the quintessence of mathematics.

Finally, Ladrière and Zubiri share an important aspect in their respective philosophy: the historicity and fragility of mathematical reason. Before Gödel, mathematicians and philosophers thought provides an objective and exact knowledge of an ideal object. Mathematics was always in an unstoppable progress capable of total success and certainty. After the "foundational crisis" this changes dramatically. Ladrière sustains that mathematical rationality, and rationality in general, is partial, limited and historical. It is discontinuous, bound to ensure permanently its own foundation and in need to integrate its own failures. Mathematics must be in constant adaptation and control of its methods in order to arrive to some certainty. For Zubiri, reason is a search that is accomplished, realized and verified historically[113]. All knowledge is always open and limited due to human, social and historical limits. Also all knowledge is always open to be surpassed because all sketches are limited chosen from partial systems of reference[114]. As a consequence, all efforts to reduce our knowledge of reality to a particular system and sketch rest at least problematic if not impossible. History of science has shown that all reductionist projects have failed.

	Ladrière	**Zubiri**
Mathematical object	Mathematical objects are given. However, the mathematician constructs his access to it by means of objective (conceptive) concepts.	Existence and properties of a mathematical object are defined and postulated. They are freely constructed according to concepts.
Foundation of mathematics.	Mathematical reality precedes the language that allows its manifestation.	The reality of mathematical objects. They receive their reality from a system of reference.
Essence of mathematics	The essence of mathematics is a formal language defined by operations, (conceptive) concepts and rules. We reach mathematical reality by means of this formal language.	Mathematics is neither logic nor language. It is a science about a reality constructed according to concepts. We are already installed in the mathematical reality.
Interpretation of Gödel's first theorem of incompleteness.	Mathematical reality always exceeds and goes beyond the axiomatic formal language, which is partial, historical and never total. There is a horizon of mathematical reality that is always open and never fully grasped.	There are real properties that go beyond what was defined and postulated. What is constructed by postulation has "de suyo" more properties than the properties formally postulated.
Mathematical truth	Mathematical reality and its truths are objective and independent of our intellectual effort. However, we can never remove completely the uncertainties of our apprehension because we cannot represent fully mathematical reality in a closed formal system.	Mathematical truths are necessary but not eternal truths. All mathematical judgments that are truth are in conformity, but not in adequation, with mathematical reality. They are approximations to what should be a truth adequate to reality.
Q1: access to mathematical reality.	Mathematical objects manifest themselves through the formal language's representation. We develop the mathematical logos that grasps such representations.	Mathematical reality is not separated from our intellect. The separation is part of a concipient intellect not of a sentient intellect.
Q2: relation between abstract mathematics and concrete objects	Mathematical reality is autonomous with respect to perception. However, abstract mathematical representations are developed through: schematization, thematization and abstraction of general structures.	Mathematical sketch has been postulated by suggestion of field reality and at the same time is independent from it. Its independence enables a free creation of the whole content according to concepts.
Q3: how can we with a finite mind can construct infinite objects?	We do not construct the object, but its objective representation. We adapt the method to reach a diversity of objects in a historical process.	We defined and postulate real mathematical objects. Construction is a creative capacity to define and postulate an unlimited number of objects.
Q4: how we explain successful application of mathematics to the physical universe?	There is a process of interpretation and elaboration of a hypothesis that a formal system represents a physical phenomenon. This demands a process of verification.	By three postulates: (i) the cosmic reality has a mathematical structure; (ii) the creation of a sketch's content; (iii) the postulate that a sketch corresponds to a cosmic reality. Finally by testing these three postulates.

Table 6. Comparative analysis between Ladrière and Zubiri.

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Notes

1 SHAPIRO, Steward, *Thinking about mathematics. The philosophy of mathematics*, Oxford University Press, New York, 2000, p. 108.

2 Objective truth-value means that a propositions can have a value of "true" or "false".

3 SHAPIRO, p. 29.

4 "[...] arithmetic laws are analytic judgment, and therefore a priori. According to this, arithmetic would be only a further developed logic, every arithmetic theorem a logical law, albeit a derived one". FREGE, Gottlob, "The concept of number", in BENACERRAF, Paul, PUTNAM, Hilary, (Eds.), *Philosophy of mathematics. Selected Readings*, London, Cambridge University Press, 2nd Edition, 1985, p. 153.

5 Russell wrote in a note of an article the following: "[...] classes may be regarded as logical fictions, manufactured out of defining characteristics. But for the present it will simplify our exposition to treat classes as if they were real".

RUSSELL, Bertrand, "Selections from Introduction to Mathematical Philosophy", in BENACERRAF, Paul, PUTNAM, Hilary, (Eds.), *Philosophy of mathematics. Selected Readings*, London, Cambridge University Press, 2nd Edition, 1985, p. 169.

6 Quoted by Shapiro, p. 158. Hilbert himself wrote: "Qu'en serait-il de la vérité de notre savoir, de l'existence et du progrès de la science s'il n'y avait au moins en mathématique une vérité solide?" Quoted by Ladrière in

LADRIERE, Jean, *Les limitations internes des formalismes, Études sur la signification du théorème de Gödel et des theorems apparentés dans la théorie des fondements des mathématiques*, Louvain, Ed. Nauwelaerts, 1957, p. 1.

7 As Ladrière affirms, this is the platonic doctrine of the adequation of intelligence to ideas, LADRIÈRE, 1957, p. 2.

8 Hilbert himself affirmed this by the following expression: "*au commencement est le signe*". Quoted by LADRIÈRE, 1957, p. 4.

9 VON NEUMANN, Johann, "The formalist foundations of mathematics," in BENACERRAF, Paul, PUTNAM, Hilary, (Eds.), *Philosophy of mathematics. Selected Readings*, London, Cambridge University Press, 2nd Edition, 1985, p. 63.

10 Consistency of a collection of axioms is the fact that they do not contradict each other. A formalized theory $\mathbf{F}_s$ is consistent if it not possible to derive a contradiction, such A = B and A ≠ B, using the axioms and the rules of $\mathbf{F}_s$. That means that a theorem $\mathbf{T}_m$ and its negation could be proved in an inconsistent formalized system.

11 SHAPIRO, p. 149.

12 Ibid., p. 22.

13 The Second incompleteness theorem can be expressed as follows: According to the first theorem, we can formalize "*derivable in* $\boldsymbol{F}_s$" and derive in $\mathbf{F}_s$ the following proposition: (i) *If* $\boldsymbol{F}_s$ *is consistent, then* $\boldsymbol{P}_s$ *is not derivable in* $\boldsymbol{F}_s$. However we can express the proposition "$\boldsymbol{P}_s$ *is not derivable in* $\boldsymbol{F}_s$" as $\boldsymbol{P}_s^*$. Consequently we can prove the following: (ii) *If* $\boldsymbol{F}_s$ *is consistent, then* $\boldsymbol{P}_s^*$ *is derivable in* $\boldsymbol{F}_s$. This contradicts (i), which is the first incompleteness theorem. Consequently, no consistent theory can prove its own consistency. See SHAPIRO, p. 167.

14 A mathematical proposition is not provable in a system if it neither derivable from the axioms and rules of deduction nor refutable. See LADRIÈRE, 1957, p. 40.

15 SHAPIRO, p. 166.

16 As we will see, for Ladrière there is "more" truth and for Zubiri there is "more" reality.

17 KNEEBONE, G. T., *Mathematical Logic and the Foundations of Mathematics. An Introductory Survey*, Dover Publication Inc., New York, 2001, p. 247.

18 It is a law that affirms about a mathematical proposition Λ the following: either Λ or not Λ. This is normally written as $\Lambda \vee \neg\Lambda$. This is closely related to the bivalence principle that all propositions have only two possible values: "true" or "false".

19 Another intuitionist, Heyting, expresses the following: "[An intuitionist] uses language, both natural and formalized, only for communicating thoughts, i.e., to get to others or himself to follow his own mathematical ideas. Such a linguistic accompaniment is not a representation of mathematics; still less is it mathematics itself".

HEYTING, Arend, "The intuitionist foundations of mathematics," in BENACERRAF, Paul, PUTNAM, Hilary, (Eds.), *Philosophy of mathematics. Selected Readings*, London, Cambridge University Press, 2nd Edition, 1985, p. 52-53.

[20] VAN STIGT, Walter P., *Brouwer's Intuitionism,* Elsevier Science Publisher, Amsterdam, 1990, p. 173.

[21] DUBUCS, Jacques, "Constructivisme", in LECOURT, Dominique (ed.), *Dictionaire d'histoire et philosophie des sciences*, Presses Universitaires de France, Paris, 2006, p. 277.

[22] In philosophy of mathematics objects refer to numbers, points, functions, sets, geometric objects, etc.

[23] I am in debt of Shapiro's work, although I modify some elements of his typology. See Shapiro's *Thinking about mathematics. The philosophy of mathematics*.

[24] They are called Platonist because their notion of mathematical objects resembles Plato's eternal, unchangeable form that not part of the physical universe. See SHAPIRO, p. 27.

[25] SHAPIRO, p. 10.

[26] See SHAPIRO, p. 110 and NADEAU, Robert, "Logicisme", in LECOURT, Dominique (ed.), *Dictionaire d'histoire et philosophie des sciences*, Presses Universitaires de France, Paris, 2006, p. 683.

[27] Quoted in SHAPIRO, p. 206.

[28] Ibid., p. 27 and p. 205.

[29] SHAPIRO, p. 208-209.

[30] Ibid., p. 21.

[31] I use the term naturalism to refer to the position that reduces all knowledge to empirical knowledge of the physical reality. This position corresponds to what is also call empirism.

[32] SHAPIRO, p. 28.

[33] Ibid., p. 19.

[34] Ibid., p. 212.

[35] SHAPIRO, p. 25 and DUBUCS, p. 276.

[36] KNEEBONE, G.T., p. 247.

[37] However, in Heyting's formalization project of intuitionism somebody could argue that at least there is the criterion of coherence within axioms and rules.

[38] SHAPIRO, p. 175.

[39] SHAPIRO, p. 28-29.

[40] SHAPIRO, p. 26.

[41] For this type, some use the term "nominalism". I rather use conventionalism because nominalism could refer to any denial of abstract object's existence. This notion of nominalism includes as well Quine's naturalism. See LAUGIER, Sandra, ""Nominalisme moderne", in LECOURT, Dominique (ed.), *Dictionaire d'histoire et philosophie des sciences*, Presses Universitaires de France, Paris, 2006, p. 812-813.

[42] PUTNAM, Hilary, "Models and Reality" in BENACERRAF, Paul, Putnam, Hilary, (Eds.), *Philosophy of mathematics. Selected Readings*, London, Cambridge University Press, 2nd Edition, 1985, p. 443.

[43] Jean Ladrière (1921-2007) was a mathematician and philosopher born in 1921 in Nivelles, Belgium. He completed his doctoral dissertation in philosophy in 1949 after which he wrote a master's thesis about recursive functions in the field of mathematics. Finally he submitted a *thèse d'agrégation* in 1957 on the limits of formal systems, e.g. Gödel's first theorem of incompleteness and the Löwenheim-Skolem theorem, among others. His works constitute major contributions to the fields of epistemology, philosophy of science, philosophy of language, ethics, political philosophy and fundamental theology.

[44] LADRIÈRE, Jean, "La philosophie des mathématiques et le problème du formalism" in *Revue Philosophique de Louvain*, Tome 57, Louvain, Editions de l'Institut Supérieur de Philosophie, 1959, p. 618.

[45] Ladrière states: "*l'être mathématique n'est ni purement construit ni purement donné"*. Ibid., p. 615-616.

[46] Ibid., p. 614.

[47] Ibid., p. 614-615.

[48] Ibid., p. 612.

[49] LADRIÈRE Jean, *L'articulation du sens. I. Discours scientifique et parole de la foi*, Paris, Cerf, 1970, p. 64.

[50] See LADRIÈRE 1970, p. 65-66, and LADRIÈRE, 1957, p. 16.

[51] LADRIÈRE, 1970, p. 65

[52] Ibid., p. 64-65. This is Ladrière's critic to Brouwer's and Gödel's notions of intuition.

[53] LADRIÈRE, Jean, "Objectivité et réalité en mathématiques," in *Revue Philosophique de Louvain*, Tome 64, Louvain, Editions de

l'Institut Supérieur de Philosophie, 1966, p. 552.

[54] Ibid., p. 557.

[55] Ibid., p. 553.

[56] Ibid, p. 553.

[57] Ibid., p. 554.

[58] For example, the number "2" can be expressed and delimitated by different mathematical formal systems, e.g. natural numbers, integers, real numbers, complex numbers, etc.

[59] LADRIÈRE, 1966, p. 557.

[60] Ibid., p. 556.

[61] For Ladrière, there is no intuition that could give us a pre-formulated immediate grasp of the mathematical object. It is language that enables us to grasp mathematical objects in their formulations. For Ladrière, mathematical experiences refer to operations, articulations and manipulations of symbols by which we define, demonstrate and discover new objects (See LADRIÈRE, 1966, p. 559, 564). Here it is clear that Ladrière follows the classical Greek identification between the *nous* and the *logos*. This is what Zubiri calls the *logification of intellection*.

[62] Ibid., p. 562.

[63] Ibid., p. 569.

[64] Ladrière's notion of the limit of formal systems should be understood in a a positive sense: the reality of mathematical objects exceeds the representations of axiomatic formal method.

[65] LADRIÈRE, 1959, p. 621-622.

[66] LADRIÈRE, 1966, p. 564.

[67] For example, in the step of schematization, our experience with group of object leads us to the schematization of natural numbers, integers and real numbers. Then, in the thematization step, we develop real analysis from arithmetic and algebra. Finally, we develop complex analysis and abstract set theories that generalize real analysis.

[68] Sometimes the development of mathematics is stimulated by concrete physical problems. For example, the Fourier series was developed to solve the problem of heat transfer in solid bodies.

[69] LADRIÈRE, 1966, p. 573-574.

[70] Ibid., p. 575.

[71] Ladrière distinguishes, in a formal system, between the following terms: presentation, representation and interpretation (See LADRIÈRE, 1970, p. 28). The presentation of a system is the set of conventions that define it. It is the formulation of a system by means of a particular choice of symbols like the atoms ("0"), the operations ("S") and the predicates ("=") (Ibid., p. 66). The representation of a system is the correspondence established between the primitive components of the system and certain class of objects, for example symbols, numbers, ideas and concrete entities, in such a way that two different primitive components will correspond to two different objects. For example, a variable could correspond to the angle of rotation of a sphere. The interpretation of a system is a correspondence between the propositions and certain statements that are true or false independently of the system, e.g., other formal systems or some empirical statements relative to the domain of experience. Every interpretation of a formal system will put it in relation with certain domain of concrete or abstract objects. See also LADRIÈRE, 1957, p 41-43.

[72] LADRIÈRE, 1966, p. 578.

[73] Ibid., p. 550.

[74] I follow the three volumes in Spanish of "Inteligencia Sentiente": *Inteligencia y Realidad*, *Inteligencia y Logos*, and *Inteligencia y Razón*. Most of the times I follow Thomas B. Fowler's English translation *Sentient Intelligence*.

[75] ZUBIRI, Xavier, *Inteligencia y Logos*, Alianza Editorial, Madrid, 1982, p. 129. From now on I will refer to it as "IL".

[76] IL, p. 131.

[77] Ibid., p. 130.

[78] Literary characters are other examples of this type of intellectual construction.

[79] Ibid., p. 325.

[80] Ibid., p. 141.

[81] Ibid., p. 144.

[82] ZUBIRI, Xavier, *Inteligencia y Realidad*, Alianza Editorial, Madrid, 1983, p. 208. From now on I will refer to it as "IR".

[83] This is possible due to the character of sentient intelligence. The intelligence is not separated from the senses. The intelligence concipient on the contrary understand what it receives separately from the senses.

[84] IR, p. 251.

[85] For those unfamiliar with the term "field reality", I will explain briefly this concept in the context of Zubiri's three modes of intellection. Zubiri affirms there are there modes of intellection: primordial apprehension, logos and reason.

(1) *Primordial apprehension* is the primary and radical mode. It is the apprehension of the real actualized in and through itself. It is apprehending the real as real. Here one apprehends each real thing as individual. What is apprehended is actualized directly, immediately, and unitarily. In the primordial apprehension, I apprehends the real "only" in and by itself. (IL 15).

(2) The other mode of intellection is the logos (IS 12). It is a mode according to which the real is actualized not only in and through itself, but also among other things. It is a development of the primordial apprehension. I intellectively know what the real thing is in function of other realities. The *field*, also called field reality, is the *ambit of reality*, an ambit which encompasses many real things. Thus each real thing should be intellectively known therein not just in and by itself (the primordial apprehension) but also with respect to the other realities in the field (IL 15-16). The reality of each thing is intrinsically and formally open to a field. This intellection, by which I apprehend each real thing in a field, is what constitutes the *logos*. Logos is the intellection of what the real is in its reality in a field. Therefore, I intellectively know a real thing from the standpoint of other real things; I intellectively know it therefore in the field-sense (IS 275).

(3) The third mode of intellection is *reason*. I intellectively know what the real thing, not only in itself, not only in function of other realities, but also in function to the world. Reason consists in going from field reality toward worldly reality.

IS refers to Zubiri, Xavier, *Inteligencia Sentiente. Inteligencia y Realidad*, Alianza Editorial, Madrid, 1980.

[86] IR, p. 220.

[87] Ibid., p. 130.

[88] IL, p. 326.

[89] For example, (i) the natural number "3" is apprehended in reference to the system of natural numbers: {0, 1, 2, 3, …}, (ii) the triangle whose angles add 180º is apprehended in reference to the Euclidean space, (iii) the event "E_1" receives its reality from the measure space: (Ω, F, P) with sample space Ω, event space F and probability measure P. Natural numbers, the Euclidean space and the measure space (in probability theory) are three examples of systems of reference.

[90] This sketch is analogous to the process of schematization, thematization and abstraction in Ladrière.

[91] IL, p. 144.

[92] Zubiri affirms that a concept is not something primarily logical but something primarily real (IL, p. 101). Through the concept we conceive what a thing *might be* in reality. We always conceive "what" might be an apprehended thing "from" others previously apprehended. A conception is not an empty free construction. It is always suggested by other things that have been already apprehended in the field reality. This conception is first of all an abstraction. Abstraction is an intellective process by which we know one or more parts of a thing, "leaving aside" others (IL, p. 102). This abstraction is freely chosen. We can abstract a thing in a particular direction. Second, the concept is not only an abstraction, it is also a construction done by the intelligence. This construction operates over abstract properties.

[93] Let's consider a simple example, the construction, in Euclidean geometry, of a "circle" according to concepts. We will leave aside the complexity of this example in order to illustrate Zubiri's idea of "construction according to concepts". We can define a *circle* as the set of points that are equidistant from a special point, named center, in the plane. In this definition we are already following Euclidean geometry's axioms, concepts, as well as other theorems. According to Zubiri this is not the construction of the objective concept "circle". Rather it is the construction of the content of the mathematical object "circle" according to other mathematical concepts that have been already defined, constructed and apprehended, e.g., set, points, distance (or metric), plane, etc. The circle, as other geometrical figures in the Euclidean space, is suggested by our "experience" in the perceived space. However, the construction according to concepts, the content of the object "circle", is independent of the perceived space.

[94] IR, p. 127.

[95] Ibid., p. 129.

[96] Consider, for example, the three axioms proposed by Kolmogorov in probability theory: (i) The probability of an event is a non-negative real number: $P(E) \geq 0, \forall E \in \Omega$; (ii) $P(\Omega) = 1$ and $P(\varnothing) = 0$; (iii) For mutually exclusive events, $E_1, E_2, \ldots, P(E_1 \cup E_2 \cup \ldots) = \Sigma_i[P(E_i)]$. They are the real content freely chosen that postulates the reality of probability theory. They are suggested by the field reality, e.g., games of chance (die, playing cards, etc).

[97] IR, p. 252.

[98] Ibid., p. 252.

[99] Ibid., p. 253.

[100] As Zubiri interprets Gödel: "*Gödel demonstrated that what is postulated has properties which are not deducible from the postulates nor can they be logically refuted by them. The fact is, as I see it, that they are real properties of mathematical reality, and their apprehension independent of the postulates is a point in which the apprehension of reality does not coincide with logical intellection.*" IR, p. 253.

[101] "*Comprobación*" in Spanish is composed of "*com*" and "*probación*". The first part, "*com*", means "with" or "together". The second part, "*probación*", comes from "*prueba*" which means testing, proof, or verification.

[102] Zubiri affirms that experience cannot be reduced to an empirical sensible experience or perception.

[103] IR, p. 254.

[104] Consider for example in Euclidean geometry the following mathematical judgment: "the diameter is the longest chord of the circle." This conclusion has at the same time two moments: the moment of apprehension of reality and the moment of necessary truth. Here, mathematical experience is testing-together the reality of the "diameter" and the reality of "the longest cord of the circle" in the formula of "*the diameter* as being *the longest chord of the circle*".

[105] IL, p. 145-146.

[106] Ibid., p. 324.

[107] Ibid., p. 321.

[108] Ibid., p. 327.

[109] Ibid., p. 327.

[110] IR, p. 281.

[111] Ibid., p. 133.

[112] IS p. 224 and IL, p. 48.

[113] IR, p. 302-303.

[114] Ibid., p. 222.